Week 2: The combinatorial core of learning

Restriction, growth, shattering, VC dimension, and Sauer–Shelah

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UCSD · Spring 2026
DSC 190/291 Topics: Learning Theory

Today

Bridge from Week 1

Restriction and growth

Shattering and VC dimension

Sauer–Shelah lemma

The combinatorial core

What can a hypothesis class do on finitely many points?

Where we left off: online learning

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Two complexity measures appeared:

Finite classes (Halving)

$$M \leq \log_2 |\mathcal{H}|$$

Complexity = cardinality

Linear predictors (Perceptron)

$$M \leq R^2/\gamma^2$$

Complexity = inverse margin

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Is there a complexity measure that is **intrinsic to** $\mathcal{H}$ and works for **any** class?

Why $|\mathcal{H}|$ is only a proxy

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To make this concrete, let's simplify the setting.

Suppose there are only n fixed points we will ever need to predict on.

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Fix an unlabeled pool $C = \{x_1, \ldots, x_n\}$. Assume **realizability**: the labels come from some unknown $h^* \in \mathcal{H}$.

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- the adversary reveals one point from C,
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In this game, two hypotheses that agree on all points of C are **indistinguishable**.

Restriction of $\mathcal{H}$ to a pool

So the relevant object is the set of distinct labeling patterns:

$$\mathcal{H}|_C = \{(h(x_1), \ldots, h(x_n)) : h \in \mathcal{H}\} \subseteq \{0,1\}^n.$$

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We can run Halving on these patterns.

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Maintain a set $S_t \subseteq \mathcal{H}|_C$ of surviving patterns.

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Same as Halving from Week 1 — but on **labeling patterns** instead of hypotheses.

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Compare with the original Halving bound $M \leq \log_2 |\mathcal{H}|$. Now N replaces $|\mathcal{H}|$ — and N is finite even when $|\mathcal{H}| = \infty$.

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So the only possible labelings on this pool are

$$000, \quad 100, \quad 110, \quad 111.$$

Running the algorithm: thresholds

True labeling: 000 ($\theta^* < x_1$). Reveal order: x_2, x_1, x_3.

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Each mistake halves the surviving set: $4 \to 2 \to 1$.

Two mistakes — exactly $\log_2 4 = 2$. The bound is tight here.

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How many **distinct behaviors** can $\mathcal{H}$ produce on n points in the worst case?

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restriction $\rightarrow$ growth function $\rightarrow$ shattering $\rightarrow$ VC dimension $\rightarrow$ Sauer–Shelah.

Restriction and growth

Count what the class can do on a fixed finite pool

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Let $C = (x_1, \dots, x_n) \in \mathcal{X}^n$. The **restriction** of $\mathcal{H}$ to C is

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For thresholds on $x_1 < x_2 < x_3$:

$$\mathcal{H}|_C = \{000,\ 100,\ 110,\ 111\} \qquad \text{— four patterns.}$$

The growth function

Definition

For a finite set C, let $\Gamma_{\mathcal{H}}(C)$ be the number of distinct patterns in $\mathcal{H}|_C$.

The **growth function** is

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Always $1 \leq \Gamma_{\mathcal{H}}(n) \leq 2^n$.

Example 1: the unrestricted class

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Why? For any target pattern on $x_1, \ldots, x_n$, just define a function that matches it.

This is the maximum possible growth — the class has no structure at all.

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Compare: the unrestricted class has $\Gamma_{\mathcal{H}}(n) = 2^n$. Thresholds have $\Gamma_{\mathcal{H}}(n) = n + 1$. Growth captures this difference — cardinality does not ($|\mathcal{H}| = \infty$ for both).

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Another infinite class, but growth is polynomial, not exponential.

Basic properties of the growth function

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We know $\Gamma_{\mathcal{H}}(n) \leq 2^n$. When does equality hold — when does $\mathcal{H}$ realize *all* 2^n labelings?

Shattering and VC dimension

When does the class behave like all possible labelings?

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Definition

A pool $C = \{x_1, \ldots, x_n\}$ is **shattered** by $\mathcal{H}$ if $\mathcal{H}$ realizes every labeling of C:

$$\text{for every } (y_1, \ldots, y_n) \in \{0,1\}^n, \text{ there exists } h \in \mathcal{H} \text{ with } h(x_i) = y_i \; \forall\, i.$$

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On a shattered pool, the class has **complete freedom** — the learner gains nothing from knowing $\mathcal{H}$.

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- On *every* set of n points, some labelings are impossible.
- The learner always has constraints to exploit.

The breakpoint

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So there is a **breakpoint** at 1:

- Below it, $\mathcal{H}$ can still mimic the unrestricted class on the right points.
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This breakpoint is intrinsic to $\mathcal{H}$ and works for any class — exactly the complexity measure we wanted.

VC dimension

Definition

The **VC dimension** of $\mathcal{H}$, written $\mathrm{VCdim}(\mathcal{H})$, is the largest n such that some $C \subseteq \mathcal{X}$ of size n is shattered by $\mathcal{H}$.

If arbitrarily large sets can be shattered, write $\mathrm{VCdim}(\mathcal{H}) = \infty$.

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VC example: intervals

Consider intervals $\mathcal{H} = \{x \mapsto \mathbf{1}[a \le x \le b] : a, b \in \mathbb{R}\}$.

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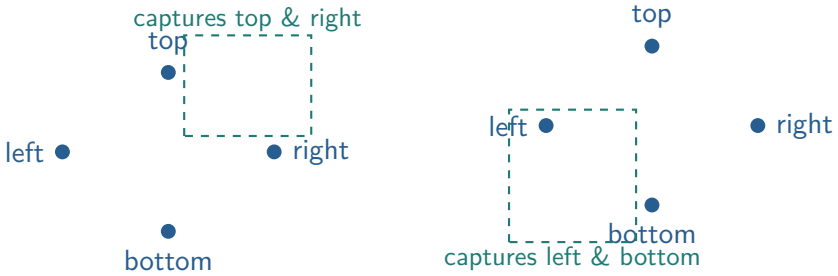
Hence $\mathrm{VCdim}(\mathcal{H}) = 2$.

VC example: rectangles — lower bound

$\mathcal{H}$ = axis-aligned rectangles in $\mathbb{R}^2$. Place 4 points in a cross:



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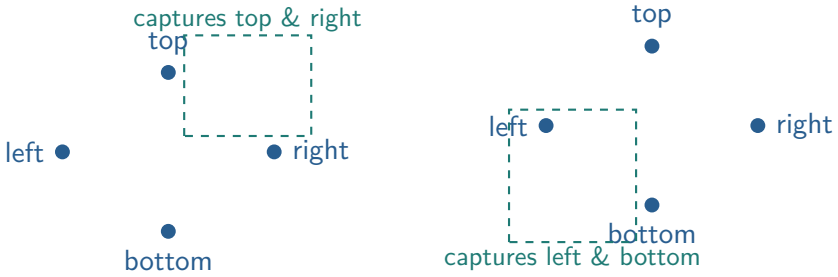

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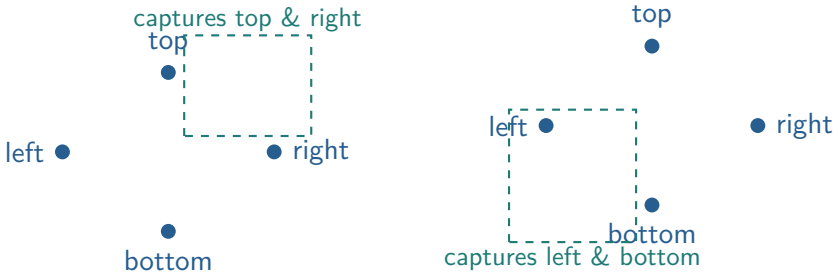
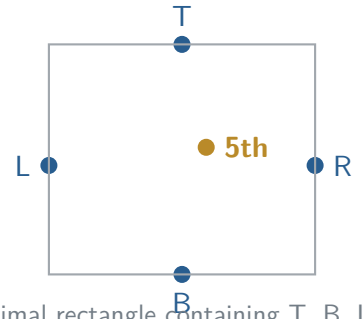


Each boundary edge independently controls one point. For any subset labeled 1, we can draw a rectangle containing exactly those points.

So these 4 points are shattered: $\mathrm{VCdim}(\mathcal{H}) \geq 4$.

VC example: rectangles — upper bound

Given any 5 points, identify the four extremals (topmost, bottommost, leftmost, rightmost):



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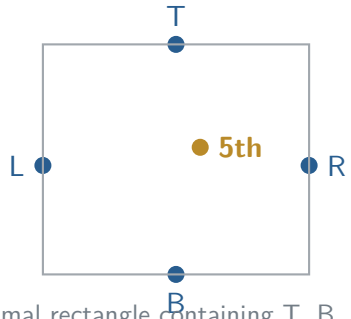

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The 5th point lies between the extremes in both coordinates — it is inside *every* rectangle that contains the other four.

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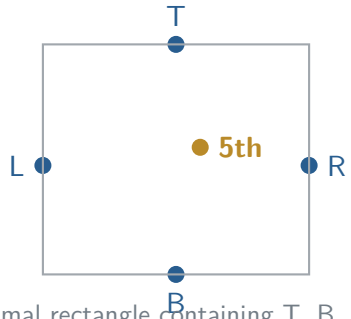

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VC example: halfspaces in $\mathbb{R}^d$

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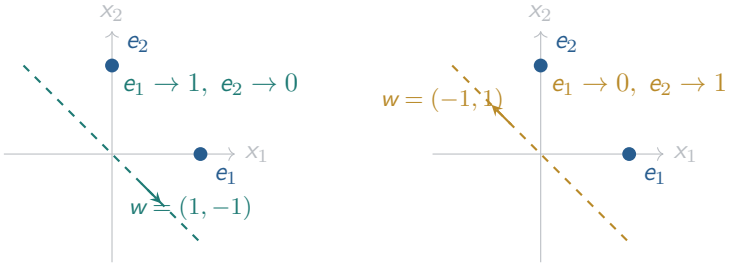
Lower bound ($\mathrm{VCdim} \geq d$): the standard basis $e_1, \ldots, e_d$ can be shattered.

For any target labeling, choose w so that $\langle w, e_i \rangle = w_i$ has the desired sign.

Halfspaces: lower bound illustrated ($\mathbb{R}^2$)

Dashed line $=$ boundary $\langle w, x \rangle = 0$. Arrow w points to the positive side ($\to 1$).



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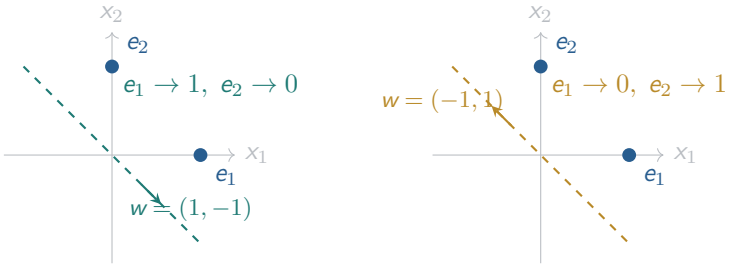

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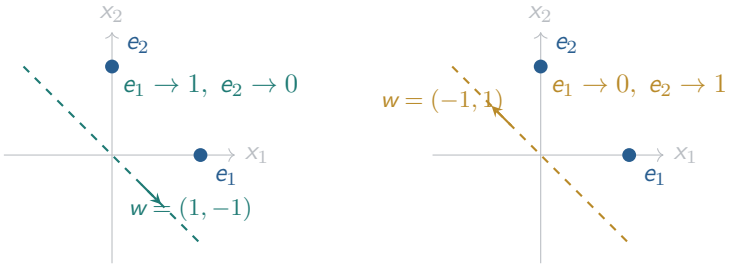

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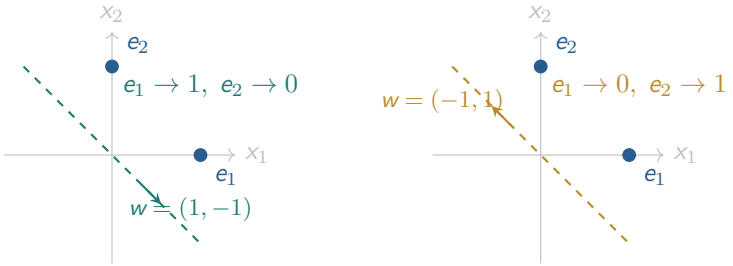

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In $\mathbb{R}^d$: same idea with d basis vectors. So $\mathrm{VCdim}(\mathcal{H}) \geq d$.

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Hence $\mathrm{VCdim}(\mathcal{H}) = d$.

The pattern in every VC computation

Every example needed two separate arguments:

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Exhibit d points that *can* be shattered.

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The upper bound is usually harder. One forbidden labeling is enough.

What VC dimension is really measuring

We've computed VCdim for several classes.

Class	VCdim
Thresholds	1
Intervals	2
Rectangles	4
Halfspaces ($\mathbb{R}^d$)	d
All functions	∞

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The largest pool size on which the class can still realize *all* labelings — giving the learner no advantage.

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Which is it?

Sauer–Shelah lemma

The key counting theorem behind VC theory

Growth beyond the VC dimension: examples

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Does $\mathrm{VCdim}(\mathcal{H}) = d$ always force $\Gamma_{\mathcal{H}}(n) = O(n^d)$?

Sauer–Shelah lemma (1972)

Yes:

Sauer–Shelah lemma

If $\mathrm{VCdim}(\mathcal{H}) = d < \infty$, then for every n,

$$\Gamma_{\mathcal{H}}(n) \leq \sum_{k=0}^{d} \binom{n}{k}.$$

In particular, for $n \geq d \geq 1$,

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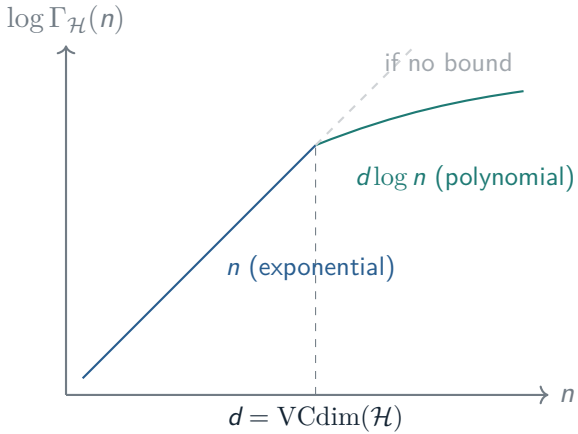
So once the class stops shattering, growth can **never** be exponential.

The phase transition in growth



At $n = d$, growth transitions from exponential to polynomial — a sharp phase transition.

Setting up the proof

Notation. Write $\Gamma_d(n)$ for the maximum of $\Gamma_{\mathcal{H}}(n)$ over all classes $\mathcal{H}$ with $\mathrm{VCdim}(\mathcal{H}) \leq d$.

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Base cases:

- $d = 0$: no single point is shattered, so every pool gets at most 1 labeling. $\Gamma_0(n) = 1 = \binom{n}{0}$.
- $n \le d$: trivially $\Gamma_d(n) \le 2^n = \sum_{k=0}^{d} \binom{n}{k}$.

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- **Type A:** only one label at x_n is compatible — extends to 1 labeling on C.
- **Type B:** both labels at x_n are compatible — extends to 2 labelings on C.

Example: prefixes for thresholds on $\{1, 2, 3\}$

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Three distinct prefixes (recall: 1s are on the left, 0s on the right):

- $(0, 0)$: $\theta < 1$, so x_3 must also be $0 \rightarrow$ **Type A**
- $(1, 0)$: $\theta \in [1, 2)$, so x_3 must be $0 \rightarrow$ **Type A**
- $(1, 1)$: $\theta \ge 2$, but x_3 depends on whether $\theta < 3$ or $\theta \ge 3 \rightarrow$ **Type B**

Example: prefixes for thresholds on $\{1, 2, 3\}$

Thresholds $\mathcal{H} = \{x \mapsto \mathbf{1}[x \le \theta] : \theta \in \mathbb{R}\}$ on $C = \{1, 2, 3\}$:

$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	prefix on $\{x_1, x_2\}$
0	0	0	$(0, 0)$
1	0	0	$(1, 0)$
1	1	0	$(1, 1)$
1	1	1	$(1, 1)$

Three distinct prefixes (recall: 1s are on the left, 0s on the right):

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Total labelings: $4 = \underbrace{3}_{\text{prefixes}} + \underbrace{1}_{\text{Type B}}$.

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Therefore $\Gamma_{\mathcal{H}}(C) \leq \Gamma_d(n-1) + b$. Bounding b is where the VC dimension enters.

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That means $\mathcal{H}$ shatters $\{x_{i_1}, \ldots, x_{i_d}, x_n\}$ — a set of $d+1$ points.
Contradiction with $\mathrm{VCdim}(\mathcal{H}) \leq d$.

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Base cases: $\Gamma_0(n) = 1$, $\Gamma_d(n) = 2^n$ for $n \le d$.
This is Pascal's rule, so by induction on n:

$$\Gamma_d(n) \le \sum_{k=0}^{d} \binom{n}{k}.$$

In particular, for $n \ge d \ge 1$,

$$\Gamma_d(n) \le \left(\frac{en}{d}\right)^d.$$

Sauer–Shelah: recap

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For any class $\mathcal{H}$ with $\mathrm{VCdim}(\mathcal{H}) = d$:

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Once the pool size n exceeds d, growth is polynomial in n, not exponential.

What this means for learning

Recall: Halving on a pool of size n makes at most $\log_2 \Gamma_{\mathcal{H}}(n)$ mistakes.

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The dominant factor is $d = \mathrm{VCdim}(\mathcal{H})$: the simpler the class, the fewer mistakes.

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Payoff. Halving + Sauer–Shelah: $O(d \log n)$ mistakes.
One number d governs learnability.

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- **Transductive setting:** given a pool, observe labels on a training subset, predict the rest.
- **Statistical (i.i.d.) setting:** data is drawn from an unknown distribution. Does VC dimension still control learning?

Project idea: formalizing the combinatorial core

Everything in this lecture forms a single chain of results:

$$\text{Restriction} \to \text{Growth function} \to \text{Shattering} \to \text{VC dimension} \to \text{Sauer–Shelah}$$

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Project: formalize this chain in Lean 4.

- Define restriction, growth function, shattering, and VC dimension.
- State and prove the Halving mistake bound: $M \leq \log_2 N$.
- State and prove Sauer–Shelah.
- Combine them: $\mathrm{VCdim}(\mathcal{H}) = d \implies M = O(d \log n)$.